Exact fields are obtained by the present method which can be used to form trial fields for variational calculations. Besides, the same method can be used for fundamental as well as other higher order modes to obtain n_e and field profiles.

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Calculation of Cutoff Wavenumbers for TE and TM Modes in Tubular Lines with Offset Center Conductor

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Abstract —**The cutoff wavenumbers of TE and TM modes (higher order modes) in a tubular line having an offset center conductor have been calculated. Whereas most previous methods used to study this structure were of an approximate nature, the analytical method developed by Singh and Kothari leads to a rigorous analytical formulation. The boundary conditions on both conductor boundaries, assumed to be perfectly conducting, are satisfied exactly. The cutoff values calculated show that some results previously reported are inaccurate.**

I. Introduction

Introducing a lateral offset in the center conductor of a coaxial line provides a simple way to decrease its characteristic impedance without modifying the dimensions of the conductors [1], [2]. This technique can be used to realize quarter-wave transformers and other matching devices. The properties of the dominant TEM mode can readily be determined with conformal mapping. Furthermore, it is also necessary to determine the cutoff frequency of higher order modes which set an upper limit to the useful frequency of operation.

The propagation along this geometry was considered by several authors [2]–[6] using approximate techniques for its analysis (in particular, point-matching and conformal mapping). While most articles did not indicate which accuracy was obtained, one recent article [6] provides an upper and a lower bound. In some instances, however, the range between the bounds is rather large (up to 20 percent), making the use of the results of little practical interest. For some other situations, the bounds for successive solutions actually overlap one another.

The same problem was tackled analytically by other authors [7]–[9]. A special perturbation method was developed in [8]. It could be useful when extended to dielectric waveguides or eccentric Goubau lines [10], but the study considers only small eccentricities. Some of the tabulated parameters of [8] actually yield nonphysical results; also, symmetric and antisymmetric modes appear to be degenerate, which contradicts experimental observations. Finally, an analytical method devised to analyze the related problem of a circular plate with an eccentric circular hole [9] yields incorrect final expressions. Detailed comments on this paper have appeared recently [17].

The analysis of previous publications shows that, even though considerable effort has been devoted to the study of this geometry, the available techniques are still either approximate when not altogether incorrect.

A rigorous mathematical derivation is presented in the present paper. The Helmholtz equation for higher order modes is solved exactly, and the boundary conditions on the two offset conductors are satisfied by the technique developed by Singh and Kothari [11], based on Graf's addition theorem for Bessel functions [12]. One obtains in this manner an infinite set of linear equations which must be truncated to permit numerical calculations. The accuracy of the results can be arbitrarily improved upon by taking additional terms.

II. Basic Theory

The longitudinal direction of the offset tubular line, to which the axes of the two conductors are parallel, is the z direction. The system is symmetrical with respect to the x axis; its transverse cross section is shown in Fig. 1, in which all significant dimensions are also reported. Two polar coordinate systems, labeled (r, θ) and (r', θ'), are defined with respect to the centers of the two conductors located at 0 and 0′, respectively.

The general solution for the transverse dependence of the potential is obtained by solving the two-dimensional Helmholtz

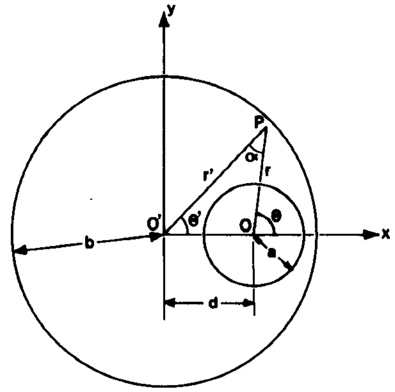

Fig. 1. Cross section of an annular eccentric circular cylindrical waveguide.

equation in the cylindrical coordinate system [13]. An infinite series expansion is obtained [14]

$$\psi(r,\theta) = \sum_{m=0}^{\infty} \{A_m J_m(kr) + B_m Y_m(kr)\}\left\{\begin{matrix}\cos\\ \sin\end{matrix} m\theta\right\} \quad (1)$$

where J_m and Y_m are the mth order Bessel functions of the first and second kind, $k = \omega\sqrt{\epsilon\mu}$ is the wavenumber in the medium between the two conductors, and A_m and B_m are constants to be determined from the boundary conditions. The cosine solution provides the symmetric modes, the sine solutions the antisymmetric ones.

For TE modes, $\psi = H_z$ must satisfy the Neumann boundary condition, while, for TM modes, $\psi = E_z$ must meet the Dirichlet boundary condition. At the edge of the inner conductor, at $r = a$, these conditions become for TE modes

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = 0, \quad \text{hence } A_m J_m'(ka) + B_m Y_m'(ka) = 0$$

and for TM modes

$$\psi(a) = 0, \quad \text{hence } A_m J_m(ka) + B_m Y_m(ka) = 0 \quad (2)$$

where the prime denotes the derivative with respect to the function's argument.

The boundary condition on the outer conductor must now be satisfied at $r' = b$, i.e., within the polar coordinate system related to 0′. The coordinate transformation from one system to the other is taken care of by Graf's addition theorem for Bessel functions [12]

$$Z_m(kr)\exp(jm\alpha) = \sum_{p=-\infty}^{p=\infty} Z_{m+p}(kr')J_p(kd)\exp(jp\theta') \quad (3)$$

where $Z_m(x)$ stands either for $J_m(x)$ or $Y_m(x)$, $\alpha = \theta - \theta'$ (Fig. 1) and $r' > d$. The expression (1) can then be transformed into the primed coordinate system; the boundary conditions—similar to (2)— are introduced at $r' = b$, yielding a set of linear equations the determinant of which must vanish if nontrivial solutions are to exist.

For TE modes

$$\det|P_{mn}(b)| = 0 \quad (4)$$

where the elements of the determinant are given by

$$P_{mn}(b) = \left(\frac{J_n'(kb)}{J_m'(ka)} - \frac{Y_n'(kb)}{Y_m'(ka)}\right) \times \left(J_{n-m}(kd) + (-1)^l J_{n+m}(kd)\right) \quad (5)$$

TABLE I
TE CUTOFF WAVENUMBERS OF ECCENTRIC WAVEGUIDES FOR VARIOUS η AND D

	SYMMETRIC			ANTISYMMETRIC		
	Present Method	Lower Bound	Upper Bound	Present Method	Lower Bound	Upper Bound
η = 2/3, d = 0.2	1.2522★	1.32027	1.32221	1.1917	1.19001	1.19176
	2.4365	2.4408	2.4448	2.4307★	2.4267	2.4305
	3.6209	3.6157	3.6281	3.6203	3.6142	3.6203
	4.7897	4.7804	4.7901	4.7896	4.7804	4.7899
	5.9379	5.9218	5.9385	5.9379	5.9231	5.9385
η=0.475, d=0.315	1.4407★	1.5132	1.5199	1.3740	1.3715	1.3741
	2.7256	2.7210	2.7345	2.7187	2.7125	2.7198
	3.9240	3.8978	3.9248	3.9244	3.9069	3.9247
	-	4.343	4.407	5.0796	5.041	5.083
	5.0799	4.977	5.087	5.3686	5.325	5.427
η = 1/3, d = 2/9	1.5619★	1.5766	1.5807	1.5435	1.5393	1.5476
	2.9064	2.8968	2.9067	2.9058	2.8966	2.9067
	4.1152	4.0944	4.1161	4.1152	4.0955	4.1161
	4.4220★	4.2146	4.2356	5.1606	5.131	5.167
	5.2669	5.219	5.270	5.2758	5.237	5.279
η = 0.5, d = 0.2	1.3793★	1.40694	1.40793	1.3522	1.35114	1.35219
	2.6849	2.6837	2.6882	2.6838	2.5815	2.6840
	3.9295	3.9247	3.9298	3.9296	3.9247	3.9298
	-	4.9937	5.0192	5.1131	5.1015	5.1138
	5.1131	5.1031	5.1139	5.8106	5.793	5.834
η = 0.25, d = 0.25	1.6650★	1.6768	1.6813	1.6490	1.6446	1.6490
	2.9678	2.9445	2.9684	2.9667	2.9547	2.9682
	-	3.939	3.983	4.1579	4.120	4.160
	4.1191	4.109	4.163	5.0616★	4.973	5.060
	5.2942	5.111	5.303	5.2965	5.175	5.302
η = 0.15875, d=0.379	1.7769	1.7603	1.7946	1.7584	1.7330	1.7584
	2.9932	2.871	3.004	2.9848	2.873	2.989
	3.8632★	3.432	3.775	4.1590	3.78	4.17
	4.1808	3.76	4.21			

Lower and upper bounds are from [6].

For TM modes

$$\det|Q_{mn}(b)| = 0 \quad (6)$$

with the elements of the determinant

$$Q_{mn}(b) = \left(\frac{J_n(kb)}{J_m(ka)} - \frac{Y_n(kb)}{Y_m(ka)}\right) \times \left(J_{n-m}(kd) + (-1)^l J_{n+m}(kd)\right). \quad (7)$$

In both (5) and (7), $l = m$ for symmetric modes and $l = m + 1$ for antisymmetric modes.

The limiting case of the coaxial line is obtained by simply letting $d = 0$ within (5) and (7). All the off-diagonal terms of the matrices (4) and (6) then vanish, yielding $P_{mm}(b) = 0$ and $Q_{mm}(b) = 0$, which are precisely the dispersion equations for the TE and TM modes in a coaxial line [15]. This fact actually provides a simple way to check the accuracy of the calculations by taking their limit when $d \to 0$ and comparing with the values in [16].

III. Results and Discussion

Results are normalized by letting the radius b of the outer cylinder equal unity. The zeros of (4) and (6) were determined, for several values of the eccentricity d and of the radius ratio η, in all cases for the first five TE and TM modes, both symmetrical and antisymmetrical. Numerical values accurate to the fifth decimal place were obtained, with determinants up to order 10. In practice, such a high level of accuracy is not required since the asymmetry of the conductors, irregularities in the cross sections, finite conductivity of the metallic boundaries, and imperfect dielectric may well introduce significant disturbances. In the present case, highly accurate values were calculated in order to

TABLE II
TE CUTOFF WAVENUMBERS OF ECCENTRIC WAVEGUIDES FOR VARIOUS η AND D

	SYMMETRIC			ANTISYMMETRIC		
	Present Method	Lower Bound	Upper Bound	Present Method	Lower Bound	Upper Bound
η = 0.5, d = 0.1	5.4695	5.46911	5.4704	5.9918	5.99121	5.99251
	6.4747	6.47403	6.47547	6.9203	6.91953	6.92102
	7.3062	7.30527	7.30683	7.7123	7.71130	7.71299
	7.8692	7.86823	7.86982	8.4845	8.4830	8.4857
	8.4965	8.4950	8.4972	9.3564	9.3542	9.3572
η = 0.5, d = 0.2	4.8106	4.80953	4.81191	5.5114	5.5098	5.5125
	6.1724	6.1703	6.1735	6.7991	6.7964	6.8002
	7.3945	7.3907	7.3957	7.9607	7.9559	7.9619
	8.4974	8.4894	8.4991	9.0091	8.9996	9.0106
	9.3409	9.2694	9.3485	9.9556	9.9316	9.9677
η = 0.5, d = 0.3	4.3071	4.3042	4.3118	5.1222	5.1179	5.1257
	5.8903	5.8730	5.8940	6.6210	6.5994	6.6251
	7.3197	7.240	7.325	7.9910	7.878	7.997
	8.2909	8.081	8.316	9.1877	8.829	9.210
	8.6388	8.382	8.648	9.2676	8.900	9.276
η = 2/3, d = 0.2	6.2399	6.2379	6.2429	6.9683	6.9654	6.9702
	7.6769	7.6728	7.6787	8.3682	8.3631	8.3800
	9.0439	9.0323	9.0456	9.7053	9.6922	9.7071
	10.3536	10.318	10.356	10.9892	10.947	10.992
	11.6184*	11.539	11.616	12.2266*	12.128	12.219
η = 0.25, d = 0.25	3.4723	3.4687	3.4752	4.2640	4.2583	4.2680
	4.9221	4.9110	4.9249	5.5393	5.5239	5.5423
	5.9268	5.393	5.932	6.6357	6.502	6.941
	6.7154	6.591	6.723	7.7135	7.443	7.723
	6.7527	6.622	6.767	7.7243	7.488	7.735
η = 0.25, d = 0.5	2.9824	2.887	2.996	4.0338	3.858	4.043
	4.7868	4.088	4.827	5.5432	4.50	5.75
	5.8084		5.977	6.9144		6.992
	6.2439		6.323	7.1560		7.208
	7.5592		7.735	8.1858		8.395

Lower and upper bounds are from [6].

allow for comparison with the results previously published by Kuttler [6].

Table I presents the symmetric and antisymmetric cutoff wavenumbers in terms of η and d for TE modes, while the corresponding values for TM modes appear in Table II. Both tables also list the lower and upper bounds provided by Kuttler [6]. In most cases, the actual cutoff wavenumber lies close to Kuttler's upper bound, while in some cases (marked by an asterisk), they fall outside of the bounds. Furthermore, in three situations marked by a bar (—), it was not possible to obtain a zero of the determinant within or near the ranges reported by Kuttler, even when taking determinants of order 15.

All the calculations were made with perfect conductor boundaries, i.e., the effect of metallic losses was not taken into account. As the values of the fields can be determined, the effect of finite metal conductivity on the cutoff may be evaluated using the usual perturbation technique. It should, however, be noted that the cutoff wavenumber then becomes complex, so that the cutoff is not clearly defined as in the lossless case.

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Fully Computer-Aided Synthesis of a Planar Circulator

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I. INTRODUCTION

A planar junction circulator consists, in general, of a three-fold symmetric resonator of arbitrary shape to which three transmission lines are connected. So far, the circulators having disk [1], triangle, and hexagonal resonators [2], [3] have been studied in detail. In the analysis of the circuit parameters of circulators, two general methods which were presented in 1977 [4] are used widely. One is based upon a contour-integral solution of the wave equation. In the other approach, the circuit parameters of the junction are expanded in terms of the eigenmodes of the magnetized planar resonator. Since both methods have been applied successfully to various circulators, there is no doubt as to the usefulness of the methods at present.

As a next step, we have to develop an algorithm to synthesize a circulator. We had studied the optimum design of a planar circulator for wide-band operation based upon a computer-aided, but trial-and-error, approach [5]. The results obtained were far

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